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FURTHER INVESTIGATION ON THE EXACT ELASTICITY SOLUTION FOR ANISOTROPIC THICK RECTANGULAR PLATES

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Abstract—The state equation governing the anisotropic response of monoclinic linear elastic material is deduced. An exact three-dimensional (3-D) elasticity solution is performed for simply-supported thick orthotropic rectangular plates subjected to arbitrary loading. For the first time, a sixth-order differential equation governing the transverse displacement is obtained and various expressions of the solution are provided. The results can be used to the plate of any kind of reduced material properties for example transversely isotropic or isotropic plates. © 1997 Elsevier Science Ltd

INTRODUCTION

Plate structures are usually analyzed by employing approximate two-dimensional (2-D) theories based on either the classical Kirchhoff–Love hypothesis of straight inextensional normals, or refinements to it to include the effects of transverse shear deformation and normal stretch. However, to assess the validity of these approximate theories, rigorous analytical solutions based on the exact three-dimensional (3-D) theory of elasticity should be obtained for some plate problems, for example anisotropic plate and thick plate. Such elasticity solutions are very valuable, especially for laminated composite structures in which the inherent anisotropic inhomogeneity lead to a complicated coupling effect, and to abrupt variations of the stresses at the interface of the laminate.

There are relatively few exact solutions to the full equations of 3-D elasticity theory as applied to the deformation of anisotropic plates. The first problem to be considered was that of cylindrical bending of a simply-supported orthotropic strip under sinusoidal transverse load (Pagano, 1969). Later investigations considered the cases of finite rectangular plates (Pagano, 1970). The problem of bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates was solved by Srinivas and Rao (1970). In a recent development, by adapting work by Iyengar and Pandya (1984) that used a state space transfer matrix method, Fan and Ye (1990a, b) have derived solutions in the form of double Fourier series, appropriate for rectangular plates. Rogers *et al.* (1992) have provided the solution process for anisotropic elliptical plate of moderate thickness, but no numerical results were available. They also gave an exact 3-D solution for the deformation and stress distribution in a semi-infinite strip clamped along its two edges (Rogers *et al.*, 1995). Nevertheless, these exact solutions have enabled us to quantify the errors involved in various plate theories.

However, two more questions which relate with the application of the aforementioned solutions exist besides the limitation of geometric and boundary conditions. One is the solution given by Fan and Ye (1990a) and Srinivas and Rao (1970), which cannot be used for the plate problem with reduced material properties. For example, transversely isotropic and isotropic materials because it is possible that the eigenvalue of eigen eqn (13) has repeated roots. A much more complicated solution process should be outlined when eqn (13) has repeated roots. Another question is about the value of H in formula (16). Pagano

(1970) said it is possible that the situation $H > 0$ cannot exist in real materials, although the heavy algebra involved would render the proof of this statement extremely difficult. Thus, this is only the case where $H < 0$ is considered in that paper (Pagano, 1970). This opinion has considerable influence on the following similar investigations performed by other researchers. More than two decades passed and still some authors hold such kinds of ideas (Bhaskar and Varadan, 1993). In fact, Pagano also pointed out that the sign of H must be investigated for the specific material and geometric properties in his paper (Pagano, 1970). Our investigation of this paper demonstrates that H can be any real number (positive, negative or zero) depending on the combination of the material and geometric properties and value of ζ, η shown in eqn (11). In order to clarify the statement above and quantify the errors analysis compared with appropriate plate theories, we shall further investigate these problems, which are very important for the programme being developed in the future, based on so-called exact solution.

NOTATION AND STATE EQUATION

To generalize the illustration, we consider the boundary value problem of 3-D elasticity. A rectangular Cartesian coordinate x, y, z is used. The constitutive law of the material for monoclinic linear elasticity can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}. \tag{1}$$

The strain–displacement relations and the equilibrium equations are

$$\varepsilon_x = \partial U/\partial x, \quad \varepsilon_y = \partial V/\partial y, \quad \varepsilon_z = \partial W/\partial z, \quad \gamma_{xy} = \partial U/\partial y + \partial V/\partial x,$$
$$\gamma_{yz} = \partial V/\partial z + \partial W/\partial y, \quad \gamma_{zx} = \partial W/\partial x + \partial U/\partial z \tag{2}$$

and

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = \rho \dfrac{\partial^2 U}{\partial t^2} \\[2ex] \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = \rho \dfrac{\partial^2 V}{\partial t^2} \\[2ex] \dfrac{\partial \tau_{xz}}{\partial z} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = \rho \dfrac{\partial^2 W}{\partial t^2} \end{cases}. \tag{3}$$

In the above equations, U, V and W denote the displacements along the coordinate axes x, y and z, respectively; ρ and c_{ij} are the density coefficients of the material under consideration. In general, the material behaviour is described by 13 independent moduli. We also note that if the material is orthotropic or transversely isotropic with a symmetry axis which makes an angle ϕ with the x-axis, then its stress–strain representation takes the form (1), but the 13 moduli are now all related to ϕ and to the nine or five independent moduli of the respective anisotropy.

Let $X = \tau_{xz}$, $Y = \tau_{yz}$, $Z = \sigma_z$, $\alpha = \partial/\partial x$, $\beta = \partial/\partial y$, $\xi^2 = \rho\partial^2/\partial t^2$, and eliminate membrane stresses σ_x, σ_y and τ_{xy} from eqns (1)–(3), the following state equation can be obtained.

$$\frac{\partial}{\partial z}\begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix}. \tag{4}$$

The eliminated stress components can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_2\alpha + c_7\beta & c_7\alpha + c_3\beta & -c_1 \\ c_3\alpha + c_8\beta & c_8\alpha + c_4\beta & -c_5 \\ c_7\alpha + c_6\beta & c_6\alpha + c_8\beta & -c_9 \end{bmatrix} \begin{Bmatrix} U \\ V \\ Z \end{Bmatrix}, \tag{5}$$

where $\mathbf{0}$ is a (3×3), and

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & -\alpha \\ a_{12} & a_{22} & -\beta \\ -\alpha & -\beta & \xi^2 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \xi^2 - c_2\alpha^2 - 2c_7\alpha\beta - c_6\beta^2 & -c_7\alpha^2 - (c_3 + c_6)\alpha\beta - c_8\beta^2 & c_1\alpha + c_9\beta \\ -c_7\alpha^2 - (c_3 + c_6)\alpha\beta - c_8\beta^2 & \xi^2 - c_6\alpha^2 - 2c_8\alpha\beta - c_4\beta^2 & c_9\alpha + c_5\beta \\ c_1\alpha + c_9\beta & c_9\alpha + c_5\beta & c_{10} \end{bmatrix},$$

c_i $(i = 1, 2, \ldots, 10)$ and a_{11}, a_{12}, a_{22} are all the constants related to the 13 stiffness coefficients of the material (Appendix A).

SIMPLY-SUPPORTED ORTHOTROPIC RECTANGULAR PLATES

For an orthotropic body, the state equation (4) can be simplified as follows:

$$\frac{\partial}{\partial z}\begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & a_{11} & 0 & -\alpha \\ 0 & 0 & 0 & 0 & a_{22} & -\beta \\ 0 & 0 & 0 & -\alpha & -\beta & \xi^2 \\ \xi^2 - c_2\alpha^2 - c_6\beta^2 & -(c_3 + c_6)\alpha\beta & c_1\alpha & 0 & 0 & 0 \\ -(c_3 + c_6)\alpha\beta & \xi^2 - c_6\alpha^2 - c_4\beta^2 & c_5\beta & 0 & 0 & 0 \\ c_1\alpha & c_5\beta & c_{10} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix}. \tag{6}$$

Similarly,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_2\alpha & c_3\beta & -c_1 \\ c_3\alpha & c_4\beta & -c_5 \\ c_6\beta & c_6\alpha & 0 \end{bmatrix} \begin{Bmatrix} U \\ V \\ Z \end{Bmatrix}. \tag{7}$$

Equations (6) and (7) are exactly the formulas (1) and (2) in the Fan and Ye's paper (1990a). The boundary conditions may be specified as (Fig. 1):

$$\text{on } x = 0 \text{ and } a; \quad \sigma_x = 0, \quad W = 0 \text{ and } V = 0,$$

$$\text{on } y = 0 \text{ and } b; \quad \sigma_y = 0, \quad W = 0 \text{ and } U = 0. \tag{8}$$

Let

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} U_{mn}(z)\cos(m\pi x/a)\sin(n\pi y/b) \\ V_{mn}(z)\sin(m\pi x/a)\cos(n\pi y/b) \\ W_{mn}(z)\sin(m\pi x/a)\sin(n\pi y/b) \end{bmatrix} e^{i\omega_{mn}t}. \tag{9}$$

The substitution of eqn (9) and eqn (6) gives

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} X_{mn}(z)\cos(m\pi x/a)\sin(n\pi y/b) \\ Y_{mn}(z)\sin(m\pi x/a)\cos(n\pi y/b) \\ Z_{mn}(z)\sin(m\pi x/a)\sin(n\pi y/b) \end{bmatrix} e^{i\omega_{mn}t}. \tag{10}$$

From eqns (7), (9) and (10), it can be noted that the boundary conditions of (8) are satisfied.

Substituting eqns (9) and (10) into state equation (6) gives the following results for each combination of m and n:

$$\frac{d}{dz}\begin{Bmatrix} U_{mn} \\ V_{mn} \\ Z_{mn} \\ X_{mn} \\ Y_{mn} \\ W_{mn} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & a_{11} & 0 & -\zeta \\ 0 & 0 & 0 & 0 & a_{22} & -\eta \\ 0 & 0 & 0 & \zeta & \eta & -\rho\omega^2 \\ -\rho\omega^2 + c_2\zeta^2 + c_6\eta^2 & (c_3+c_6)\zeta\eta & c_1\zeta & 0 & 0 & 0 \\ (c_3+c_6)\zeta\eta & -\rho\omega^2 + c_6\zeta^2 + c_4\eta^2 & c_5\eta & 0 & 0 & 0 \\ -c_1\zeta & -c_5\eta & c_{10} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ Z_{mn} \\ X_{mn} \\ Y_{mn} \\ W_{mn} \end{Bmatrix}, \tag{11}$$

where

$$\zeta = m\pi/a, \quad \eta = n\pi/b, \quad \omega = \omega_{mn}.$$

From eqn (11), a sixth-order differential equation governing any of the six components ($U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}, Z_{mn}$) can be obtained. For example, the equation governing the transverse displacement W_{mn} is

$$\frac{d^6 W_{mn}}{dz^6} + A_0 \frac{d^4 W_{mn}}{dz^4} + B_0 \frac{d^2 W_{mn}}{dz^2} + C_0 W_{mn} = 0. \tag{12}$$

A_0, B_0 and C_0 can be determined from the coefficient matrix of eqn (11) (Appendix B). If W_{mn} in eqn (12) is known, all other stress and displacement components ($U_{mn}, \ldots, X_{mn}, \ldots$, etc.) can easily be determined from eqns (11) and (7). We note that only a fourth-order differential equation governing the transverse displacement is available in classical plate theory. So far, the sixth-order differential equation governing transverse displacement is first given for 3-D (orthotropic) plate problem (eqn (12)). It is also true for the monoclinic material plate with any boundary conditions, the only difference is that A_0, B_0 and C_0 are functions of operators α, β and elastic coefficients c_{ij}.

SOLUTION

Assuming the solution of eqn (12) in the form $W_{mn} = ke^{\lambda z}$ (k is an arbitrary constant), the characteristic equation of eqn (12) is given by

$$\lambda^6 + A_0\lambda^4 + B_0\lambda^2 + C_0 = 0. \tag{13}$$

Equation (13) can be transformed into a third-degree equation as

$$\gamma^3 + p\gamma + q = 0, \tag{14}$$

where $\gamma = \lambda^2 - A_0/3$, and

$$p = B_0 - A_0^2/3, \quad q = C_0 + A_0(2A_0^2/9 - B_0)/3. \tag{15}$$

The nature of the solution of (14) is controlled by the sign of the quantity H, where

$$H = p^2/4 + q^3/27. \tag{16}$$

Pagano (1970) only considered the case with $H < 0$. It is incomplete. Numerical results show that H can be either positive or negative even for the plate with the same material and geometric properties. For transversely isotropic or isotropic material, H can be zero. Thus, various expressions of the solution for eqn (12), depending on the value of H, will be given here to complete the solution we discussed.

(1) $H > 0$. Making notation $f_1 = \sqrt[3]{-q/2 + \sqrt{H}}, f_2 = \sqrt[3]{-q/2 - \sqrt{H}}$ (obviously $f_1 > f_2$), the solution will be

$$\begin{aligned} W_{mn} = {} & k_1 sh\mu_1 z + k_2 ch\mu_1 z + sh(\mu_2 \cos(\phi/2)z)[k_3 \sin(\mu_2 \sin(\phi/2)z) + k_4 \cos(\mu_2 \sin(\phi/2)z)] \\ & + ch(\mu_2 \cos(\phi/2)z)[k_5 \sin(\mu_2 \sin(\phi/2)z) + k_6 \cos(\mu_2 \sin(\phi/2)z)], \quad (f_1 + f_2 - A_0/3 > 0). \end{aligned} \tag{17}$$

$$\begin{aligned} W_{mn} = {} & k_1 \sin \mu_1 z + k_2 \cos \mu_1 z + sh(\mu_2 \cos(\phi/2)z)[k_3 \sin(\mu_2 \sin(\phi/2)z) + k_4 \cos(\mu_2 \sin(\phi/2)z)] \\ & + ch(\mu_2 \cos(\phi/2)z)[k_5 \sin(\mu_2 \sin(\phi/2)z) + k_6 \cos(\mu_2 \sin(\phi/2)z)], \quad (f_1 + f_2 - A_0/3 < 0). \end{aligned} \tag{18}$$

$$\begin{aligned} W_{mn} = {} & k_1 + k_2 z + sh(\mu_2 \cos(\phi/2)z)[k_3 \sin(\mu_2 \sin(\phi/2)z) + k_4 \cos(\mu_2 \sin(\phi/2)z)] \\ & + ch(\mu_2 \cos(\phi/2)z)[k_5 \sin(\mu_2 \sin(\phi/2)z) + k_6 \cos(\mu_2 \sin(\phi/2)z)], \quad (f_1 + f_2 - A_0/3 = 0). \end{aligned} \tag{19}$$

In which $\mu_1 = \sqrt{|f_1 + f_2 - A_0/3|}$, $\mu_2 = \sqrt[4]{3(f_1 - f_2)^2/4 + (A_0/3 + f_1/2 + f_2/2)^2}$ and

$$\phi = \begin{cases} \theta & (A_0/3 + f_1/2 + f_2/2 < 0) \\ \pi - \theta & (A_0/3 + f_1/2 + f_2/2 > 0) \\ \pi/2 & (A_0/3 + f_1/2 + f_2/2 = 0) \end{cases} \quad \text{and} \quad \theta = \text{arctg}\frac{\sqrt{3}(f_1 - f_2)}{|f_1 + f_2 + 2A_0/3|}, \quad (0 < \theta < \pi/2).$$

(2) $H < 0$. Making notation

$$f_3 = 2\sqrt{-p/3}\cos((2\pi - \phi)/3) - A_0/3, \quad f_4 = 2\sqrt{-p/3}\cos((2\pi + \phi) - A_0/3,$$

then

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_1 z + k_3 sh\mu_2 z + k_4 ch\mu_2 z + k_5 sh\mu_3 z + k_6 ch\mu_3 z, \quad (f_3 > 0, f_4 > 0). \tag{20}$$

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_1 z + k_3 \sin\mu_2 z + k_4 \cos\mu_2 z + k_5 \sin\mu_3 z + k_6 \cos\mu_3 z, \quad (f_3 < 0, f_4 < 0). \tag{21}$$

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_1 z + k_3 sh\mu_2 z + k_4 ch\mu_2 z + k_5 \sin\mu_3 z + k_6 \cos\mu_3 z, \quad (f_3 > 0, f_4 < 0). \tag{22}$$

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_1 z + k_3 sh\mu_2 z + k_4 ch\mu_2 z + k_5 + k_6 z, \quad (f_3 > 0, f_4 = 0). \tag{23}$$

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_1 z + k_3 + k_4 z + k_5 \sin\mu_3 z + k_6 \cos\mu_3 z, \quad (f_3 = 0, f_4 > 0), \tag{24}$$

where $\mu_1 = \sqrt{2\sqrt{-p/3}\cos(\phi/3) - A_0/3}$, $\mu_2 = \sqrt{|f_3|}$, $\mu_3 = \sqrt{|f_4|}$ and

$$\phi = \begin{cases} \theta & (q < 0) \\ \pi - \theta & (q > 0); \\ \pi/2 & (q = 0) \end{cases} \quad \theta = \text{arctg}\frac{2\sqrt{-H}}{|q|}, \quad (0 < \theta < \pi/2).$$

(3) $H = 0$. Making notation $f_5 = -2\sqrt[3]{q/2} - A_0/3, f_6 = 2\sqrt[3]{q/2} - A_0/3$, the solution will be

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_2 z + k_3 sh\mu_2 z + k_4 ch\mu_2 z + k_5 z \cdot sh\mu_2 z + k_6 z \cdot ch\mu_2 z, \quad (f_5 > 0, f_6 > 0). \tag{25}$$

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_1 z + k_3 \sin\mu_2 z + k_4 \cos\mu_2 z + k_5 z \cdot \sin\mu z + k_6 z \cdot \cos\mu_2 z, \quad (f_5 > 0, f_6 > 0). \tag{26}$$

$$W_{mn} = k_1 \sin\mu_1 z + k_2 \cos\mu_1 z + k_3 sh\mu_2 z + k_4 ch\mu_2 z + k_5 z \cdot sh\mu_2 z + k_6 z \cdot ch\mu_2 z, \quad (f_5 < 0, f_6 > 0). \tag{27}$$

$$W_{mn} = k_1 sh\mu_1 z + k_2 ch\mu_1 z + k_3 + k_4 z + k_5 z^2 + k_6 z^3, \quad (f_5 > 0, f_6 = 0). \tag{28}$$

$$W_{mn} = k_1 + k_2 z + k_3 sh\mu_2 z + k_4 ch\mu_2 z + k_5 z \cdot sh\mu_2 z + k_6 z \cdot ch\mu_2 z, \quad (f_5 = 0, f_6 > 0). \tag{29}$$

$$W_{mn} = k_1 sh\mu_3 z + k_2 ch\mu_3 z + k_3 z \cdot sh\mu_3 z + k_4 z \cdot ch\mu_3 z + k_5 z^2 \cdot sh\mu_3 z + k_6 z^2 \cdot ch\mu_3 z, \quad (q = 0), \tag{30}$$

where $\mu_1 = \sqrt{|f_5|}$, $\mu_2 = \sqrt{|f_6|}$ and $\mu_3 = \sqrt{-A_0/3}$.

From formulas (17)–(30), there are 14 kinds of solutions for eqn (12), depending on the sign of H. Of course, not all solutions will be applied for a special problem. Perhaps only one or several solutions exist. Moreover, the solutions provided here can be applied to the transversely isotropic and isotropic plates because it is possible that repeated roots will occur in both cases. In fact, formula (30) is the solution with three repeated pairs of roots in eqn (12), and is suitable to the isotropic plate (Pagano, 1970).

NUMERICAL RESULTS

Six integral constants k_i $(i = 1, 2, \ldots, 6)$ in the solution of eqn (12) can be determined according to the force vectors applied to the external plate plane $(z = 0, h)$. For example, if the upper and the lower surfaces of a rectangular plate are subjected to arbitrary normal load $q_0(x, y)$ and $q_1(x, y)$, respectively, the boundary conditions can be written as

$$\begin{Bmatrix} X_{mn}(0) \\ Y_{mn}(0) \\ Z_{mn}(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{4}{ab}\displaystyle\int_0^a\int_0^b q_0(x,y)\sin(\zeta x)\sin(\eta y)\mathrm{d}x\,\mathrm{d}y \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} X_{mn}(h) \\ Y_{mn}(h) \\ Z_{mn}(h) \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ \dfrac{4}{ab}\displaystyle\int_0^a\int_0^b q_1(x,y)\sin(\zeta x)\sin(\eta y)\mathrm{d}x\,\mathrm{d}y \end{Bmatrix}. \tag{31}$$

Because $X_{mn}(z)$, $Y_{mn}(z)$ and $Z_{mn}(z)$ can be expressed by W_{mn} and its derivative from eqn (11), the six integral constants k_i $(i = 1, 2, \ldots, 6)$ in W_{mn} can be uniquely determined by eqn (31).

Example 1

A rectangular orthotropic thick plate with simply-supported edges (Fig. 1) is subjected to normal load $-0.5q_0\sin(\pi x/a))\sin(\pi y/b)$ on its upper surface $(z = 0)$ and $0.5q_0\sin(\pi x/a)$ $\sin(\pi y/b)$ on its lower surface $(z = h)$, i.e. an antisymmetric loading about $z = h/2$ is applied. The boundary conditions are

$$\begin{Bmatrix} X(0) \\ Y(0) \\ Z(0) \end{Bmatrix} = \begin{Bmatrix} X_{11}(0) \\ Y_{11}(0) \\ Z_{11}(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -q_0/2 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} X(h) \\ Y(h) \\ Z(h) \end{Bmatrix} = \begin{Bmatrix} X_{11}(h) \\ Y_{11}(h) \\ Z_{11}(h) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_0/2 \end{Bmatrix}. \tag{32}$$

Numerical calculations are performed with the following values

$$E_x = 10E_y, \quad E_y = E_z, \quad G_{xy} = G_{xz} = 0.6E_z,$$
$$G_{yz} = 0.5E_z, \quad \mu_{xy} = \mu_{xz} = \mu_{yz} = 0.25, \quad a = b. \tag{33}$$

It can easily be verified from eqns (15), (16) and Appendix A that $H > 0$

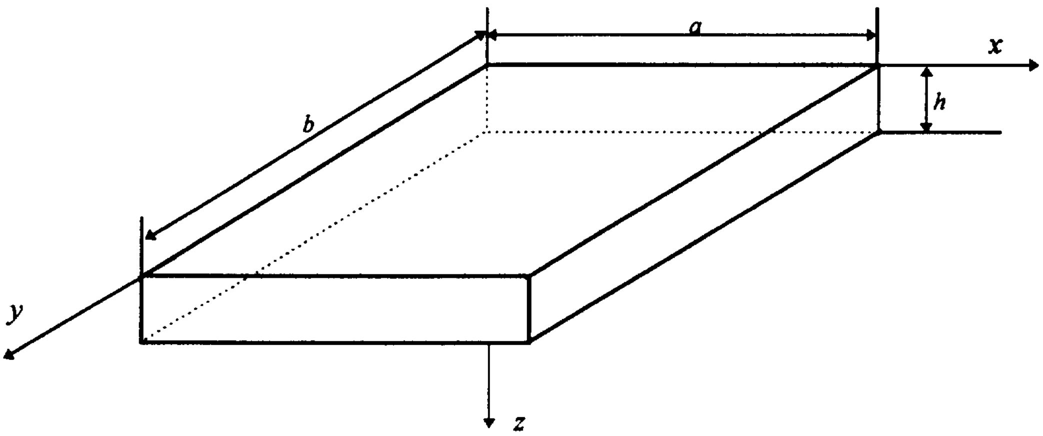

Fig. 1. Coordinate system and plate dimensions.

Table 1. Variations of maximum membrane stress components across thickness

Theories		Ambartsumyan			Present three-dimensional solution		
h/a	z/h	σ_x/q_0 $x = a/2$, $y = b/2$	σ_y/q_0 $x = a/2$, $y = b/2$	τ_{xy}/q_0 $x = 0$, $y = 0$	σ_x/q_0 $x = a/2$, $y = b/2$	σ_y/q_0 $x = a/2$, $y = b/2$	τ_{xy}/q_0 $x = 0$, $y = 0$
	0.0	−11.6390	−1.9457	1.2314	−11.5818	−1.9394	1.5732
	0.1	−8.3860	−1.5195	0.8598	−8.3721	−1.5157	1.1858
	0.2	−5.7498	−1.1180	0.5717	−5.7838	−1.1162	0.8487
0.2	0.3	−3.5762	−0.7350	0.3463	−3.6291	−0.7345	0.5470
	0.4	−1.7110	−0.3644	0.1627	−1.7482	−0.3644	0.2680
	0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0	−5.4810	−1.1206	0.4688	−5.4148	−1.1152	0.8135
	0.1	−3.5080	−0.8547	0.2488	−3.4824	−0.8484	0.5758
0.3	0.2	−2.1196	−0.6166	0.1129	−2.1686	−0.6117	0.3916
	0.3	−1.1695	−0.3995	0.0402	−1.2521	−0.3964	0.2431
	0.4	−0.5117	−0.1963	0.0096	−0.5713	−0.1948	0.1164
	0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0	−2.4258	−0.6136	0.0298	−2.3629	−0.6165	0.3845
	0.1	−1.1519	−0.4405	−0.1043	−1.0964	−0.4309	0.2251
0.5	0.2	−0.4039	−0.3010	−0.1530	−0.4746	−0.2894	0.1290
	0.3	−0.0501	−0.1866	−0.1376	−0.1849	−0.1778	0.0695
	0.4	0.0407	−0.0891	−0.0795	−0.0589	−0.0845	0.0302
	0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2. Variations of maximum displacement components across thickness

Theories		Ambartsumyan			Present three-dimensional solution		
h/a	z/h	$U \cdot E_z/(q_0h)$ $x = 0$, $y = b/2$	$V \cdot E_z/(q_0h)$ $x = a/2$, $y = 0$	$W \cdot E_z/(q_0h)$ $x = a/2$, $y = b/2$	$U \cdot E_z/(q_0h)$ $x = 0$, $y = b/2$	$V \cdot E_z/(q_0h)$ $x = a/2$, $y = 0$	$W \cdot E_z/(q_0h)$ $x = a/2$, $y = b/2$
	0.0	1.7551	2.4346	8.8725	1.7463	2.4268	8.8233
	0.1	1.2554	1.8986	8.8725	1.2534	1.8921	8.8422
	0.2	0.8549	1.3930	8.8725	0.8605	1.3906	8.8493
0.2	0.3	0.5787	0.9145	8.8725	0.5373	0.9136	8.8503
	0.4	0.2519	0.4530	8.8725	0.2580	0.4528	8.8494
	0.5	0.0000	0.0000	8.8725	0.0000	0.0000	8.8488
	0.0	0.5386	0.9110	2.5256	0.5317	0.9070	2.5505
	0.1	0.3370	0.6886	2.5256	0.3346	0.6837	2.5371
0.3	0.2	0.1980	0.4930	2.5256	0.2036	0.4889	2.5191
	0.3	0.1060	0.3175	2.5256	0.1151	0.3146	2.5027
	0.4	0.0452	0.1554	2.5256	0.0517	0.1541	2.4914
	0.5	0.0000	0.0000	2.5256	0.0000	0.0000	2.4875
	0.0	0.1367	0.2725	0.6211	0.1327	0.2753	0.6887
	0.1	0.0588	0.1870	0.6211	0.0556	0.1833	0.6572
0.5	0.2	0.0146	0.1221	0.6211	0.0197	0.1172	0.6259
	0.3	−0.0043	0.0728	0.6211	0.0048	0.0689	0.6006
	0.4	−0.0064	0.0338	0.6211	0.0003	0.0318	0.5845
	0.5	0.0000	0.0000	0.6211	0.0000	0.0000	0.5789

($H = 1385.1(\pi/a)^{12}$) and $f_1 + f_2 - A_0/3 > 0$. Hence, the solution expression (17) exists. Because it is an antisymmetric structure, Tables 1 and 2 only give stress and displacement components of half the thickness.

It is seen that most maximum stress and displacement components between Ambartsumyan theory (1970) and 3-D elasticity agree well when h/a is small. However, the Ambartsumyan theory cannot give a correct prediction for shear stress τ_{xy}. The discrepancy increases with the increasing of ratio h/a. When h/a equals 0.5, the stresses and displacements given by Ambartsumyan theory are incredible (even the sign of some values has changed)!

Table 3. Calculation values of H based on the material properties of formula (33)

m	1	3	5	7	9	11	13	15	17	19
n						H $[(10\pi/a)^{12}]$				
1	1.4×10^{-9}	-3.2×10^{-5}	-0.029	-1.854	-39.526	-4.5×10^{2}	-3.4×10^{3}	-1.9×10^{4}	-8.5×10^{4}	-3.3×10^{5}
3	1.4×10^{-7}	7.4×10^{-4}	0.052	0.132	-16.753	-2.9×10^{2}	-2.5×10^{3}	-1.5×10^{4}	-7.3×10^{4}	-2.9×10^{5}
5	5.0×10^{-6}	4.8×10^{-3}	0.338	5.866	41.704	1.0×10^{2}	-5.9×10^{2}	-7.7×10^{3}	-4.7×10^{4}	-2.1×10^{5}
7	1.2×10^{-4}	0.020	1.127	19.178	1.6×10^{2}	8.5×10^{2}	2.9×10^{3}	5.8×10^{3}	-2.6×10^{3}	-8.2×10^{4}
9	1.8×10^{-3}	0.074	3.006	46.680	3.9×10^{2}	2.2×10^{3}	8.8×10^{3}	2.7×10^{4}	6.6×10^{4}	1.1×10^{5}

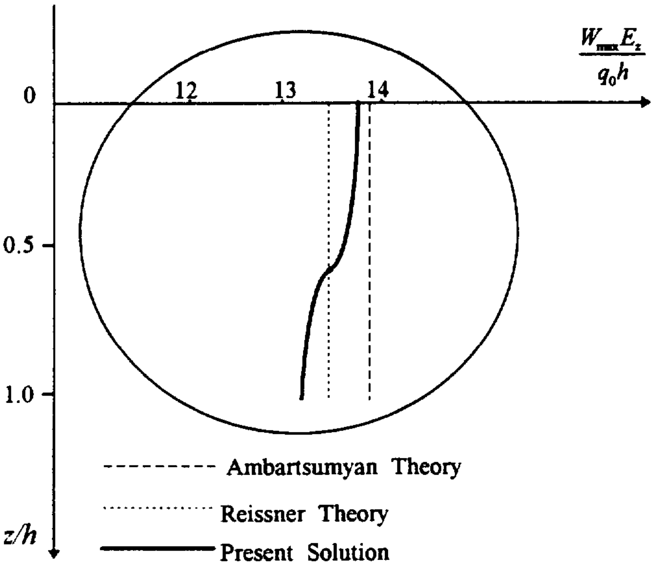

Fig. 2. Maximum transverse displacement across thickness of the plate ($(h/a) = 0.2$) loaded on its upper surface ($z = 0$) by a uniform normal loading q_0.

Example 2

All elastic constants and geometric properties are the same as that of Example 1. The plate is only loaded by a uniform loading q_0 on its upper plane ($z = 0$). Here we list some values of H and find that either positive or negative sign of H is possible depending on m and n (Table 3). The corresponding solutions are, respectively, formulas (17) ($H > 0$ and $f_1+f_2-A_0/3 > 0$) and (20) ($H < 0$ and $f_3 > 0, f_4 > 0$).

The summation result of 100 terms of Fourier series about x and y variables is calculated ($m, n = 1, 3, \ldots, 19$). A good convergency is guaranteed compared with the result of Fan and Ye (1990a). The maximum transverse displacement W_{max} across thickness is shown in Fig. 2. It is seen that maximum transverse displacement W_{max} changes a little across the thickness. It is more reasonable than that given by the Ambartsumyan theory (1970) and Reissner theory (1945). Moreover, it is often considered that the effect of various plate assumptions is to increase the stiffness of the structure and, therefore, yield lower deflections (Srinivas and Rao, 1970). However, a comparison in Fig. 2 does not support such an opinion especially for an anisotropic plate.

CONCLUDING REMARKS

The main contributions in this paper are:

(1) A state equation of 3-D elasticity of monoclinic linear elastic body is deduced.

(2) A sixth-order differential equation governing transverse displacement W, compared with the fourth-order one in classical plate theory, is given for the first time.

(3) Various possible solutions for simply-supported orthotropic rectangular plate are provided, which can be applied to the transversely isotropic and isotropic cases.

(4) Numerical results demonstrate that H in formula (16) can be any real number. The sign of H, which determines the nature of the solution, depends on the combination of material, geometric and loading properties in a practical problem.

(5) Some remarks on the difference in results between several plate theories and exact 3-D theory are also given.

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APPENDIX A

Elastic constants in state equation (4)

$$c_1 = -c_{13}/c_{33},\quad c_2 = c_{11}-c_{13}^2/c_{33},\quad c_3 = c_{12}-c_{13}c_{23}/c_{33},\quad a_{11} = c_{44}/(c_{44}c_{55}-c_{45}^2),$$

$$c_4 = c_{22}-c_{23}^2/c_{33},\quad c_5 = -c_{23}/c_{33},\quad c_6 = c_{66}-c_{36}^2/c_{33},\quad a_{12} = -c_{45}/(c_{44}c_{55}-c_{45}^2),$$

$$c_7 = c_{16}-c_{13}c_{36}/c_{33},\quad c_8 = c_{26}-c_{23}c_{36}/c_{33},\quad c_9 = -c_{36}/c_{33},\quad a_{22} = c_{55}/(c_{44}c_{55}-c_{45}^2),$$

$$c_{10} = 1/c_{33},$$

where c_{ij} $(i,j = 1,2,\ldots,6)$ are stiffness coefficients of monoclinic material shown in eqn (1).

For orthotropic material $c_{16} = c_{26} = c_{36} = c_{45} = 0$, thus, $c_7 = c_8 = c_9 = a_{12} = 0$. c_{ij} has a relation with engineering elastic stiffness coefficients as follows:

$$c_{11} = E_x(1-\mu_{yz}\mu_{zy})/Q,\quad c_{22} = E_y(1-\mu_{xz}\mu_{zx})/Q,\quad c_{33} = E_z(1-\mu_{xy}\mu_{yx})/Q,$$

$$c_{12} = E_x(\mu_{yx}+\mu_{zx}\mu_{yz})/Q,\quad c_{13} = E_x(\mu_{zx}+\mu_{yx}\mu_{zy})/Q,\quad c_{23} = E_y(\mu_{zy}+\mu_{xy}\mu_{zx})/Q,$$

$$c_{44} = G_{yz},\quad c_{55} = G_{zx},\quad c_{66} = G_{xy},\quad Q = 1-\mu_{xy}\mu_{yx}-\mu_{yz}\mu_{zy}-\mu_{zx}\mu_{xz}-2\mu_{xy}\mu_{yz}\mu_{zx},$$

$$E_x\mu_{yx} = E_y\mu_{xy},\quad E_x\mu_{zx} = E_z\mu_{xz},\quad E_y\mu_{zy} = E_z\mu_{yz}.$$

The subscripts of E and G are the Young's modulus and shear modulus at given directions. μ_{xy} is Poisson's ratio which characterizes the contraction (expansion) in the direction of the y-axis during tension (compression) in the direction of the x-axis, and so forth.

APPENDIX B

Coefficients in the governing equation (12)

Taking notation $\zeta = m\pi/a$, $\eta = n\pi/b$ and

$$b_1 = (-c_1+c_{10}/a_{11})\zeta,$$

$$b_2 = (-c_5+c_{10}/a_{22})\eta,$$

$$k_{11} = a_{11}[(c_2+c_1^2/c_{10})\zeta^2+c_6\eta^2-\rho\omega_{mn}^2],$$

$$k_{22} = a_{22}[(c_4+c_5^2/c_{10})\eta^2+c_6\zeta^2-\rho\omega_{mn}^2],$$

$$k_{33} = c_{10}(\zeta^2/a_{11}+\eta^2/a_{22}-\rho\omega_{mn}^2),$$

$$k_{12} = a_{11}(c_3+c_6+c_1c_5/c_{10})\zeta\eta,$$

$$k_{21} = a_{22}(c_3+c_6+c_1c_5/c_{10})\zeta\eta,$$

then, A_0, B_0 and C_0 will be

$$A_0 = -k_{11}-k_{22}-k_{33}+b_1^2a_{11}/c_{10}+b_2^2a_{11}/c_{10},$$

$$B_0 = k_{11}k_{22}+k_{22}k_{33}+k_{33}k_{11}-k_{12}k_{21}-b_1^2k_{22}a_{11}/c_{10}-b_2^2k_{11}a_{22}/c_{10}+2b_1b_2k_{12}a_{22}/c_{10},$$

$$C_0 = -k_{33}(k_{11}k_{22}-k_{12}k_{21}),$$

where ρ is the density of the material. The frequency of harmonic oscillations ω_{mn} is shown in eqn (9). For static problem, $\omega_{mn} = 0$.